ECE 174 – Lecture Supplement – SPRING 2009

Review of Gaussian Elimination and LU Factorization

Ken Kreutz–Delgado
Electrical and Computer Engineering
Jacobs School of Engineering
University of California, San Diego

VERSION ECE174LSGE–S09v1

April 7, 2009

The Elementary Row Matrices (ERM's) $E_{p,q}(\alpha)$ and $P_{p,q}$

Elementary row matrices are defined by the *elementary row operations* which they perform.

- $E_{p,q}(\alpha)$: Add $\alpha \cdot \text{row}_p$ to row_q.
  I.e., $\text{row}_q := \text{row}_q + \alpha \cdot \text{row}_p$

  $P_{p,q}$: Interchange (Permute) rows p and q.

- $E^{-1}_{p,q}(\alpha) = E_{p,q}(-\alpha)$, $E_{p,q}(\alpha)$ is lower triangular for $p < q$.

$$E_{2,3}(4) = E_{2,3}(4)\, I_3 = E_{2,3}(4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

- $P_{p,q} = P^{-1}_{p,q} = P_{q,p}$, $P^2_{p,q} = I$.

$$P_{1,3} = P_{1,3}\, I_3 = P_{1,3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

It is important to note that ERM's *premultiply* the matrix that they are operating one. Also note that $P_{p,q} = P^T_{p,q}$, which is true *only* for an *elementary* permutation matrix. It is *not true* for a *general* permutation matrix P. A (general) permutation matrix P is defined to be a product of elementary permutation matrices,

$$P \triangleq P_{p,q} \dots P_{k,l}$$

and has the property that

$$P^{-1} = P^T \Leftrightarrow PP^T = I.$$

As mentioned, in general $P \neq P^T = P^{-1}$.

Example 1. Gaussian Elimination (GE).

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 4}$$

$$U \triangleq \underbrace{E_{2,3}(-2)\, E_{1,3}(1)\, E_{1,2}(-2)}_{\triangleq\, \hat{\boldsymbol{L}}}\, A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Recall that:

- The matrix $U = \hat{L}A$ is said to be in *Upper Echelon Form.*
- Because $\hat{L}$ is a product of lower triangular matrices, it is itself lower triangular.
- Pivots are 1 and 3. (Pivots cannot have the value 0.)
- *Rank* of $A \triangleq r(A) \triangleq$ Number of Pivots.
- $r(A)$ = Number of linearly independent columns = dim $\mathcal{R}(A)$.
- The pivot columns of A are linearly independent and form a *basis* for $\mathcal{R}(A)$.
- $r(A)$ = Number of linearly independent rows = dim $R(A^T) \Rightarrow$ dim $R(A)$ = dim $R(A^T)$.

$$A = E_{1,2}(2)E_{1,3}(-1)E_{2,3}(2)\, U = L\, U \quad \text{where}$$

$$L \triangleq E_{1,2}(2)E_{1,3}(-1)E_{2,3}(2) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

- Note that $L = \hat{L}^{-1}$. The inverse of a lower (respectively, upper) triangular invertible matrix is always a lower (upper) triangular matrix.
- Note the pattern which holds between the elements of L and its factors $E_{p,q}(\alpha)$. (This pattern does *not* hold for the elements of $\hat{L}$).

■■■

Example 2. GE with Row Exchange.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}, \quad \mathrm{r(A)} = 3$$

$$P_{2,3}\, E_{1,3}(-2)\, E_{1,2}(-1)\, A = U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Longrightarrow \; E_{1,2}(-2)\, E_{1,3}(-1)\, P_{2,3}\, A = U$$

$$\Longrightarrow \; \underbrace{P_{2,3}A}_{\triangleq\, \boldsymbol{A'}} = E_{1,3}(1)\, E_{1,2}(2)\, U = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

■■■

LU Factorization

We have shown that there is an equivalence between Gaussian elimination (which you first encounter in middle school) and *LU factorization.* Without loss of generality, one often discusses the simpler problem $A = LU$. This is because one can always "fix" a matrix A for which this is not true via the transformation $A \leftarrow A' = PA$, where P is a product of elementary permutation matrices which rearranges the rows of A.

We often want an even simpler structure. Namely, we would like the pivots of A to be on the *main diagonal* of U. Thus U of example 2 is OK in this regard, while U of example 1 can be placed into the simpler diagonal form by permuting *columns* 2 and 3. This natural leads us to a discussion of *elementary column matrices.*

Elementary Column Matrices

Postmultiplication of a matrix A by an elementary matrix results in an *elementary column operation.* In particular postmultiplication a matrix A by the elementary permutation matrix $P_{p,q}$ results in a swapping of column p with column q. As before, $P_{p,q}^{-1} = P_{p,q} = P_{p,q}^T$.

Example 1 continued.

$$\hat{L}\, A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{L}\,A\,P_{2,3} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\underbrace{A\,P_{2,3}}_{A'} = L\,U.$$

Thus the transformed matrix A' has an $L\,U$ factorization where L is lower triangular and U is upper echelon with pivots on the diagonal.

■■■

Most generally, if we premultiply a matrix A from the left by a permutation matrix P_L (to rearrange the rows) and postmultiply from the right by a permutation matrix P_R (to rearrange the columns) we can always place A into a form A' which has an $L\,U$ factorization with the pivots on the diagonal of U,

$$A' = P_L\,A\,P_R = L\,U\,.$$

For ease of exposition, and without loss of generality, in most discussions of LU factorization it is common to assume the simpler case that $A = L\,U$, where L is lower triangular and U is upper echelon with pivots on the diagonal. This is because one can always "fix" A to ensure that this is true via the transformation $A \leftarrow A' = P_L\,A\,P_R$.

Solving the Linear Inverse Problem $Ax = b$

The same row operations $\hat{L}$ acting on both sides of the equation $Ax = b$ preserves equality,

$$Ax = b \implies \hat{L}Ax = \hat{L}b\,.$$

The simultaneous operation of $\hat{L}$ on A and b can be written in the equivalent form

$$(A \quad b) \implies \hat{L}\,(A \quad b) = (\hat{L}\,A \quad \underbrace{\hat{L}\,b}_{\triangleq\, c}\,)$$

Example 1 continued.

With $\hat{L} = E_{2,3}(-2)\,E_{1,3}(1)\,E_{1,2}(-2)$ then

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \implies c = \hat{L}\,b = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}$$

Alternatively, one can find the value of c by solving the system $L\,c = b$ using *forward substitution.* Once the value of c has been determined, we can then focus on the system $\hat{L}A\,x = c$. Thus

$$\underbrace{\hat{L}A\,P_{2,3}}_{U}\;\underbrace{P_{2,3}^T\,x}_{\bar{x}} = c$$

$$P_{2,3}^T\, x = P_{2,3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} = \bar{x}$$

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 3 & 3 & 2 \\ 0 & 3 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]}_{U} \underbrace{\left[\begin{array}{c} \bar{x}_1 \\ \bar{x}_2 \\ \hline \bar{x}_3 \\ \bar{x}_4 \end{array}\right]}_{\bar{x}} = c \tag{1}$$

$$\underbrace{\left[\begin{array}{c|c} U_1 & U_2 \\ \hline 0 & 0 \end{array}\right]}_{U} \underbrace{\left[\begin{array}{c} \bar{x}_b \\ \hline \bar{x}_f \end{array}\right]}_{\bar{x}} = c$$

with

$$U_1 = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}, \quad \bar{x}_b = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \quad \text{and} \quad \bar{x}_f = \begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}.$$

Note the following about the above:

- The r pivot columns are the *first* r columns of U which we assemble into the matrix U_1. The remaining columns of U are assembled into the matrix U_2.
- The components of $\bar{x}_b$, $\bar{x}_1$ and $\bar{x}_2$ (i.e. the original components x_1 and x_3), are known as *basic variables.* They correspond to the columns of U (and the columns of A) *with pivots.*
- The components of $\bar{x}_f$, $\bar{x}_3$ and $\bar{x}_4$ (i.e. x_2 and x_4), are known as *free variables.* They correspond to the columns of U (and columns the columns of A) *without pivots.*

■■■

More generally, for an arbitrary $m \times n$ matrix A of rank r we have

$$P_L\, A\, P_R = L\, U = L \left[\begin{array}{c|c} U_1 & U_2 \\ \hline 0 & 0 \end{array}\right] = L \left[\begin{array}{cccc|ccc} p_1 & \times & \cdots & \times & \times & \cdots & \times \\ 0 & p_2 & \cdots & \times & \times & \cdots & \times \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & p_r & \times & \cdots & \times \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right]$$

where

$$U_1 = \begin{bmatrix} p_1 & \cdots & \times \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_r \end{bmatrix}$$

is an upper-triangular and invertible $r \times r$ matrix with the r (nonzero) pivots on the diagonal. We also have

$$\bar{x} = \begin{bmatrix} \bar{x}_b \\ \bar{x}_f \end{bmatrix} = P_R^T \, x$$

where the components of $\bar{x}_b \in \mathbb{R}^r$ are the *basic variables* and the components of $\bar{x}_f \in \mathbb{R}^n$ are the *free variables.* Thus under the action of a succession of elementary matrix operations (which we commonly referred to as Gaussian Elimination) we obtain

$$Ax = b \quad \Longrightarrow \quad \underbrace{\hat{L}\, P_L \, A \, P_R}_{\mathbf{U}} \; \underbrace{P_R^T \, x}_{\bar{\boldsymbol{x}}} = \underbrace{\hat{L}\, P_L b}_{\mathbf{c}}$$

so that $Ax = b$ has been transformed into the equivalent system

$$U\bar{x} = c \qquad \Longleftrightarrow \qquad \left[\begin{array}{c|c} U_1 & U_2 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} \bar{x}_b \\ \hline \bar{x}_f \end{array}\right] = \left[\begin{array}{c} c_u \\ \hline c_\ell \end{array}\right].$$

Note that by partitioning the matrix $\hat{L} = L^{-1}$ as $\hat{L} = \binom{\hat{L}_u}{\hat{L}_\ell}$ the vector $c = \hat{L}P_L b$ has the structure

$$c = \begin{bmatrix} c_u \\ c_\ell \end{bmatrix} = \begin{bmatrix} \hat{L}_u \\ \hat{L}_\ell \end{bmatrix} P_L b = \begin{bmatrix} \hat{L}_u P_L b \\ \hat{L}_\ell P_L b \end{bmatrix}$$

so that

$$c_u = \hat{L}_u P_L b \quad \text{and} \quad c_\ell = \hat{L} P_L b\,.$$

Lemma 1. The system $Ax = b$ has a solution (i.e., the system is *consistent*) iff $c_\ell = \hat{L}_\ell P_L b = 0$.

Proof: If $Ax = b$ has a solution, then $U\bar{x} = c$ must be consistent which implies that $c_\ell = 0$. On the other hand, if $c_\ell = 0$, then the system $U\bar{x} = c$ is consistent and $U_1 \bar{x}_b = c_u - U_2 \bar{x}_f$ can be solved for a particular solution by taking $\bar{x}_f = 0$ and solving $U_1 \bar{x}_b = c_b$ for $\bar{x}_b$ by backsubstitution. ■

Corollary 1. $b \in \mathcal{R}(A)$ iff $c_\ell = \hat{L}_\ell P_L b = 0$.

Example 1 continued.

Under a sequence of GE steps,

$$\underbrace{\hat{L} A P_{2,3}}_{\mathbf{U}} \; \underbrace{P_{2,3}^T x}_{\bar{\boldsymbol{x}}} = \underbrace{\hat{L} b}_{\mathbf{c}},$$

we have transformed the system $Ax = b$ of Example 1 into the equivalent form

$$U\bar{x} = c \qquad \Longleftrightarrow \qquad \left[\begin{array}{cc|cc} 1 & 3 & 3 & 2 \\ 0 & 3 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \bar{x}_1 \\ \bar{x}_2 \\ \hline \bar{x}_3 \\ \bar{x}_4 \end{array}\right] = \left[\begin{array}{c} c_1 \\ c_2 \\ \hline c_3 \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 - 2b_1 \\ \hline b_3 - 2b_2 + 5b_1 \end{array}\right]$$

$$\begin{bmatrix} c_1 \\ c_2 \\ \hline c_3 \end{bmatrix} = \begin{matrix} \overbrace{}^{\hat{L}_u} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \hline 5 & -2 & 1 \end{array}\right] \\ \underbrace{}_{\hat{L}_\ell} \end{matrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$c_\ell = \hat{L}_\ell b = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 5b_1 - 2b_2 + b_3$$

Therefore a solution exists iff $5b_1 - 2b_2 + b_3 = 0$ and all b vectors whose components satisfy this condition are in $\mathcal{R}(A)$.

To determine a particular solution, x_p, for a consistent system of equations, we can take $\bar{x}_f = 0$ and solve

$$U_1 \bar{x}_b = c_u$$

via backsubstitution. Note that $\bar{x} = P_{2,3} x \implies x = P_{2,3} \bar{x} = P_{2,3} \begin{bmatrix} \bar{x}_b \\ \bar{x}_f \end{bmatrix}$. Thus $\bar{x}_f = 0$ implies that $x_2 = 0$ and $x_4 = 0$. Also note that $\bar{x}_2 = x_3$ and $\bar{x}_1 = x_1$. Setting $\bar{x}_f = 0$ yields

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}}_{\mathbf{U_1}} \underbrace{\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}}_{\bar{\mathbf{x}}_\mathbf{b}} = \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\mathbf{c_u}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 - 2b_1 \end{bmatrix}}_{\hat{\mathbf{L}}_\mathbf{u}\mathbf{b}}$$

which can be easily solved via back substitution,

$$x_3 = \bar{x}_2 = \frac{1}{3}(b_2 - 2b_1)$$

$$x_1 = \bar{x}_1 = b_1 - 3\bar{x}_2 = b_1 - b_2\,.$$

Thus, we have obtained the particular solution

$$x = x_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 - b_2 \\ 0 \\ \frac{1}{3}(b_2 - 2b_1) \\ 0 \end{bmatrix}$$

provided that $5b_1 - 2b_2 + b_3 = 0$ For example, take $b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, then $5b_1 - 2b_2 + b_3 = -2 + 2 = 0$ and $x_p = \begin{bmatrix} -1 \\ 0 \\ \frac{1}{3} \\ 0 \end{bmatrix}$ is a particular solution for the system $Ax = b$. ■■■

Lemma 2. Let A be $m \times n$. The system $Ax = b$ has a solution for *any* vector b iff $r = \text{rank}(A) = m$.

Proof: If $r = m$, then $c_\ell \in \mathbb{R}^{m-r}$ is nonexistent and the system $U\bar{x} = b$ is always consistent. ■

Corollary 2. $\mathcal{R}(A) = \mathbb{R}^m$ iff $r = \text{rank}(A) = m$. This is also a consequence of the fact that

$$\dim \mathcal{R}(A) = r = \text{number of pivot columns.}$$

Recall that for an $m \times n$ matrix A:

- m = number of rows, n = number of columns.
- $r = \text{rank}(A)$ = number of rows with pivots $\le m$.
- $r = \text{rank}(A)$ = number of columns with pivots $\le n$.
- $r = \text{rank}(A) \le \min(m, n)$.
- when $r = m$, the matrix A is said to have *full row rank.*
- when $r = n$, the matrix A is said to have *full column rank.*
- If $r = \min(m, n)$, i.e. if A has either full row rank or full column rank, then A is *full rank.*
- If $r < \min(m, n)$, i.e. if A has neither full row rank nor full column rank, then A is said to be *rank deficient.*

Lemma 3 (Rank Test). The system $Ax = b$ has a solution iff $r(A) = r([A\ b])$.

Proof: Assume, with no loss of generality, that $\hat{L}A = U$, $A = LU$. Then

$$\hat{L}\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} \hat{L}A & \hat{L}b \end{bmatrix} = \begin{bmatrix} U & c \end{bmatrix} = \left[\begin{array}{c|c|c} U_1 & U_2 & c_u \\ \hline 0 & 0 & c_\ell \end{array}\right].$$

Note that

$$r(A) = r(U) = \text{number of nonzero rows of } U$$

while

$$r([A\ b]) = r([U\ c]) = \text{number of nonzero rows of } [U\ c]\,.$$

Then $r(A) = r\begin{bmatrix} A & b \end{bmatrix}$ iff $c_\ell = 0$ which, in turn, is true iff $Ax = b$ has a solution by Lemma 1. ■

Characterization of the Nullspace $\mathcal{N}(A)$

Up to a possible need to permute columns and/or rows of a matrix, we have seen that Gaussian elimination results in a system in the partitioned form

$$\underbrace{\left[\begin{array}{c|c} U_1 & U_2 \\ \hline 0 & 0 \end{array}\right]}_{\hat{L}A = U} \left[\begin{array}{c} x_b \\ \hline x_f \end{array}\right] = \underbrace{\left[\begin{array}{c} c_u \\ \hline c_\ell \end{array}\right]}_{\hat{L}b = c}$$

Note that

$$x \in \mathcal{N}(A) \iff Ax = LUx = 0 \iff Ux = 0 \iff x \in \mathcal{N}(U)$$

yielding the very useful result that

$$\mathcal{N}(A) = \mathcal{N}(U)\,.$$

This means that if we can characterize the vectors in the nullspace of U (i.e., determine those vectors x for which $Ux = 0$) then we have characterized the vectors in the nullspace of A (i.e., those vectors x for which $Ax = 0$). The condition for $x \in \mathcal{N}(U)$ is

$$Ux = \left[\begin{array}{c|c} U_1 & U_2 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} x_b \\ \hline x_f \end{array}\right] = 0 \iff U_1 x_b = -U_2 x_f \iff x_b = -U_1^{-1} U_2 x_f$$

Thus,

$$\begin{aligned} x \in \mathcal{N}(A) &\iff x \in \mathcal{N}(U) \\ &\iff U_1 x_b = -U_2 x_f \\ &\iff x_b = -U_a^{-1} U_2 x_f \\ &\iff x = \begin{bmatrix} x_b \\ x_f \end{bmatrix} = \begin{bmatrix} -U_1^{-1} U_2 x_f \\ x_f \end{bmatrix} = \begin{bmatrix} -U_1^{-1} U_2 \\ I \end{bmatrix} x_f = N x_f \end{aligned}$$

where

$$N = \begin{bmatrix} -U_1^{-1} U_2 \\ I \end{bmatrix} \in \mathbb{R}^{n \times \nu}$$

$$v = n - r = \dim \mathcal{N}(A)\,.$$

The dimension of the nullspace of A, ν, is known as the nullity of A and is given by $n - \text{rank}(A)$.

The ν columns of the matrix N are linearly independent and span the nullspace of A. Thus the columns of A form a basis for $\mathcal{N}(A)$. Every nullspace vector x must be of the form $x = N x_f$ showing that the ν components of the ν-dimensional vector x_f completely parameterizes the nullspace of A.

As mentioned, the columns of N, n_k, $k = 1, \cdots, \nu$, provide a basis for $\mathcal{N}(A)$. We can mathematical determine n_k as follows. Define the *canonical basis vector* e_k by

$$e_k = (0 \cdots 0\ 1\ 0 \cdots 0)^T$$

which is the vector with all zero components except for the value "1" for the k-th component. Now note that taking $x_f = e_k$ yields $x = N e_k = n_k \in N(A)$.

$$n_k = N e_k = \begin{bmatrix} -U_1^{-1} U_2\, e_k \\ e_k \end{bmatrix} = \begin{bmatrix} n_k^{(u)} \\ e_k \end{bmatrix} \Rightarrow U_1\, n_k^{(u)} = -U_2\, e_k$$

Example 1 continued.

Recalling a column permutation has been performed, we have $x \in \mathcal{N}(A)$ iff $\bar{x} \in \mathcal{N}(U)$, which is true iff

$$\begin{bmatrix} U_1 & U_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_b \\ \bar{x}_f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives the condition

$$\left[\begin{array}{cc|cc} 1 & 3 & 3 & 2 \\ 0 & 3 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \bar{x}_1 \\ \bar{x}_2 \\ \hline \bar{x}_3 \\ \bar{x}_4 \end{array}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which, in turn, implies that

$$U_1 \bar{x}_b = -U_2 \bar{x}_f \quad \Longleftrightarrow \quad \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = - \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}.$$

To determine $\nu = n - 4 = 4 - 2 = 2$ null spaces basis vectors, we set $\bar{x}_f$ equal to e_1 and e_2 as follows,

$$\bar{x}_f = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \overline{n}_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow n_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{x}_f = e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \overline{n}_2 = \begin{pmatrix} -1 \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \Rightarrow n_2 = \begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

It is easily checked that $An_1 = An_2 = 0$. The two basis vectors form the columns of the matrix $N \in \mathbb{R}^{n \times \nu} = \mathbb{R}^{4 \times 2}$,

$$N = \begin{bmatrix} n_1 & n_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & 0 \\ 0 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix}.$$

Note that a general nullspace vector $x_0 \in \mathcal{N}(A)$ has the form

$$x_0 = N\bar{x}_f = x_2\, n_1 + x_4\, n_2\,.$$

Whenever the consistency condition

$$c_f = \hat{L}_f b = 5b_1 - 2b_2 + b_3 = 0$$

is satisfied, then $b \in \mathcal{R}(A)$ and the system $Ax = b$ has the particular solution x_p determined above. In this case, a general solution is given by

$$x = x_p + x_0 = \underbrace{\begin{bmatrix} b_1 - b_2 \\ 0 \\ \frac{1}{3}(b_2 - 2b_1) \\ 0 \end{bmatrix}}_{\boldsymbol{x_p}} + \underbrace{x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}}_{\boldsymbol{x_0 = N\bar{x}_f = x_2 n_1 + x_4 n_2}}$$

Note that we have an uncountable infinity of possible solutions. ■■■

Lemma 4. Let A be $m \times n$ and $b \in \mathcal{R}(A)$. Then exists a *unique* solution to the system $Ax = b$ iff $r = n$.

Proof: $r = n \iff \nu = n - r = 0 \iff \dim \mathcal{N}(A) = \nu = 0 \iff \mathcal{N}(A) = \{0\}$. ■

Corollary 4a. When A is *square*, $A \in \mathbb{R}^{n\times n}$, then the system $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$ iff $r = n$. The solution is given by $x = A^{-1}b$.

Proof: A straightforward consequence of Lemma 2 and Lemma 4, noting that $r = n = m$. ■

Corollary 4b. If $A \in \mathbb{R}^{n\times m}$ has $m < n$ then a solution to $Ax = b$ for $b \in \mathcal{R}(A)$ *must* be nonunique. Furthermore, if $r = m$ (so that A has full row rank), $Ax = b$ must be (nonuniquely) solvable for all $b \in \mathbb{R}^m$.

Concluding Remarks

Via the use of Gaussian Elimination and LU factorization applied to an $m \times n$ matrix A, one can directly determine the dimensions of all of the subspaces associated with A and basis vectors for the range and nullspace of A. One can also do the same for the matrix found by transposing A, A^T.[1]

[1] Although we do not discuss it here, one can actually obtain bases for the range and nullspace of A^T by a further processing of the LU factorization found for A itself.